Capacitance

> **Definition 0.1** (Capacitance) — The charge stored on a capacitor per unit voltage applied across it.

Essentially, capacitance is a measure of how much charge an electrical component can store per unit potential difference.
There is a linear relationship between charge stored and voltage; capacitance is simply the constant of proportionality between them.

> **Formula 0.2** (Capacitance Equation)
> $$Q = VC$$

> **Remark 0.3** The unit of capacitance is the **farad** (F), a 1 farad component can store 1 coulomb of charge at a potential difference of 1 volt.

If an object can store charge, it has a capacitance - you, the Earth, clouds *etc.*
We can find the capacitance of some simple objects, such as a sphere. A sphere of radius R with charge Q will have a radially symmetric electric field, so

$$V = \frac{Q}{4\pi\varepsilon_0 R}$$

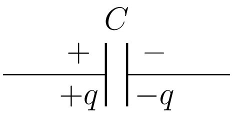
However, $C = \frac{Q}{V} = Q \times \frac{4\pi\varepsilon_0 R}{Q}$, hence

$$C_{sphere} = 4\pi\varepsilon_0 R$$

For example, the capacitance of the Earth is $4\pi\varepsilon_0 \times (6400 \times 10^3) \approx \underline{7.1 \times 10^{-4}}$, which is extremely small!

§0.1 Parallel Plate Capacitors

A parallel plate capacitor involves two surfaces, in **close proximity** to one another, which are not able to transfer current across the gap.[1]



After connecting the capacitor to a voltage supply, electrons will move in to one side of the capacitor. The electric field they generate **induces** a repulsion in the electrons on the other half of the capacitor. Providing the other half of the capacitor is also connected to the battery, electrons will leave the other plate - creating a positive charge equal to the amount of negative charge on the former.
When the voltage across the capacitor reaches the **e.m.f.** of the battery, no more current flows.
But, what is the capacitance of a parallel plate capacitor equal to? We can use the expression **??**, which we derived for the voltage between parallel plates.

$$\Delta V_{plates} = \frac{Qd}{A\varepsilon_0}, \quad C = \frac{Q}{V}$$
$$\Rightarrow C = Q \times \frac{A\varepsilon_0}{Qd} = \frac{A\varepsilon_0}{d}$$

[1] If they did, they would no longer be stores of charge, but resistors.

So, the capacitance of parallel plates of area A, a distance d apart and general electrical permittivity ε is given by

Formula 0.4 (Capacitance of Parallel Plates)

$$C = \frac{A\varepsilon}{d}$$

Example 0.5 A parallel plate capacitor has a separation of 1 mm. What area must each plate have so that the component has the same capacitance as the Earth? *[Note: $R_{Earth} \approx 6400\,km$]*

Solution. We have,

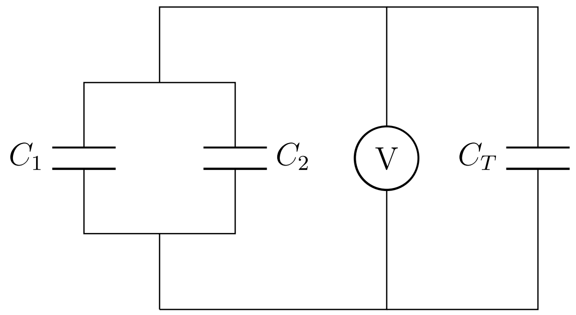
$$C_{sphere} = C_{plate} \Rightarrow 4\pi\varepsilon_0 R_E = \frac{\varepsilon_0 A}{d}$$
$$\therefore\ A = 4\pi R_E d$$
$$= 4\pi \times \left(6400 \times 10^3\right) \times \left(1 \times 10^{-3}\right) = \underline{80\,424\,\mathrm{m}^2}$$

□

§0.2 Combinations of Capacitors

As with resistors, we can combine capacitors together in certain configurations to create a component which has a different capacitance (compared to its individual parts). But how do capacitors add up?

We can consider two capacitors with capacitances C_1 and C_2. We will place them in **series** and **parallel** and treat them as an entire component, with new capacitance C_T.
We can start with case when in parallel:



Let's give the component with capacitance C_i, voltage V_i and charge Q_i.
By Kirchoff's 2nd law, it must be that $V_1 = V_2 = V_T$ and all this is equal to the voltage V read from the voltmeter.
By conservation of charge (also, Kirchoff's 1st law), $Q_T = Q_1 + Q_2$. But, recall that $Q = VC$.

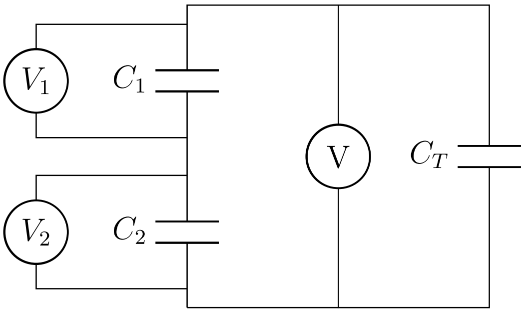
$$\therefore\ V_T C_T = V_1 C_1 + V_2 C_1$$

Though, as we established $V_T = V_1 = V_2$, these all cancel and we are left with $C_T = C_1 + C_2$. Therefore, in parallel we have the following formula.

Formula 0.6 (Capacitor Addition in Parallel)

$$C_T = \sum_i C_i$$

On the other hand, in the case for series:



By Kirchoff's 2nd law, $V = V_1 + V_2 = V_T$. Also, since current is the same across a loop in series, as per Kirchoff's 1st law, $Q_1 = Q_2 = Q_T$.
But, recall that $V = \frac{Q}{C}$.

$$\therefore \frac{Q_T}{C_T} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2}$$

Though, as we established $Q_T = Q_1 = Q_2$, these all cancel and we are left with $\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2}$. Therefore, in series we have the following formula.

Formula 0.7 (Capacitor Addition in Series)

$$\frac{1}{C_T} = \sum_i \frac{1}{C_i}$$

§0.3 Energy storage

Definition 0.8 (Dielectric) — An insulating material which prevents the conducting plates of a capacitor from coming in contact.

The **dielectric** material tends to have high resistance, but modifies the electric field. Dielectric materials **polarise**, meaning that the charge distribution of the molecules change in response to the external field.
The dielectric reduces the electric field strength between the plates by modifying the **electric permittivity** ε.

Formula 0.9 (Electric Permittivity)

$$\varepsilon = \kappa \cdot \varepsilon_0$$

where ε is the electric permittivity of the substance, ε_0 is the electric permittivity of free space and κ is known as the dielectric constant (or the relative permittivity) with $\kappa > 1$.
Capacitors can store (usually) small amounts of energy and release it over a very short time period. The energy is stored in the electric field between the plates.

As the capacitor charges, the voltage increases. The energy is represented as the **area** under the **charge-voltage graph** *i.e.* $E = \int Q \, \mathrm{d}V$.

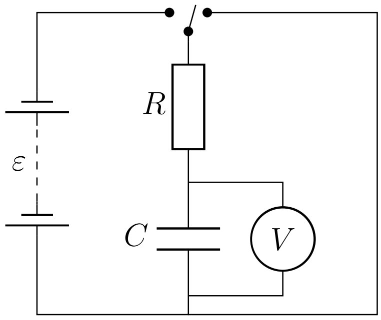
Formula 0.10 (Energy Stored in Capacitors)

$$E = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$$

Remark 0.11 Voltage is not a conserved quantity, however energy and charge are conserved quantities.

§0.4 (Dis)charging Capacitors

Suppose we wish to investigate the rate of (dis)charging of a capacitor when it is (dis)connected to a battery of emf ε. This problem is best understood when the circuit contains a resistor of resistance R, since capacitors can charge instantaneously without resistance.



When the two-way switch is such that the resistor-capacitor combination is connected to the battery:

1. Current starts to flow and charge begins to accumulate on the capacitor plates.
2. The initial voltage across the capacitor is zero (since $V = Q/C$ and $Q_0 = 0$).
3. Initially, the entire potential difference is across the resistor.
4. As the capacitor accumulates charge, the potential difference is increasingly shared with the capacitor. V across capacitor increases, V across resistor decreases.
5. When the potential difference across the capacitor matches the emf ε, the resistor has zero p.d., and current no longer flows (since $I = V_r/R$ but $V_r = 0$). The capacitor is **fully charged**.

The two-way switch is then flicked, disconnecting R-C component from the battery, and connecting the terminals of the component together.

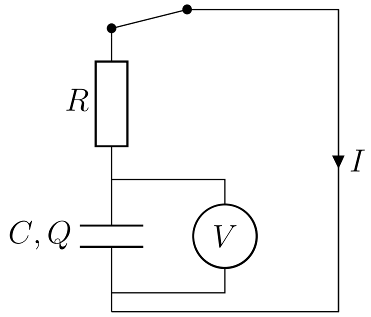
1. Electrons start leaving the negative plate of the capacitor, since they can flow through the circuit and fill the void of electrons on the positive side of the plate. The capacitor is **discharging**.
2. As voltage across the capacitor decreases with decreasing charge, so too does the voltage across the resistor - the current must start at a maximum, and is tending towards zero, until there is no charge left on the capacitor.

Remark 0.12 The absolute values of voltage across the resistor and capacitor during discharge are the same at all times.

But what does the time dependence of all these measurable quantities look like? We can answer this with some calculus!

Discharging Capacitor

Let the capacitor shown below have voltage V, charge Q and capacitance C. Let the resistor have resistance R and voltage V_R, and let the current flowing at a point in time be I.



We know by definition $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$. By Kirchoff's second law, the sum of the voltages in a closed loop is zero, so $V + V_R = 0 \Rightarrow \frac{Q}{C} + IR = 0$. This gives

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{Q}{RC} \Rightarrow \int_{Q_0}^{Q} \frac{\mathrm{d}Q}{Q} = -\frac{1}{RC}\int_0^t \mathrm{d}t$$

$$\therefore \ \ln Q - \ln Q_0 = -\frac{t}{RC} \Rightarrow \boxed{Q = Q_0 e^{-\frac{t}{RC}}}$$

Now, dividing both sides of the above equation by C gives the corresponding equation for voltage, noting that $\frac{Q_0}{C} = V_0$.

Formula 0.13 (Discharging of Capacitors - Charge and Voltage)

$$Q(t) = Q_0 e^{-\frac{t}{RC}}$$
$$V(t) = V_0 e^{-\frac{t}{RC}}$$

Recalling that $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$, we can also find $I(t)$ by differentiating.

$$\therefore \ I = -\frac{Q_0}{RC} e^{-\frac{t}{RC}} = -\frac{V_0}{R} e^{-\frac{t}{RC}} = \boxed{-I_0 e^{-\frac{t}{RC}}}$$

Formula 0.14 (Discharging of Capacitors - Current)

$$I = I_0 e^{-\frac{t}{RC}}$$

Remark 0.15 The **discharge current** has a negative sign because its direction is opposite to the **charging current**, however we often do not bother so much with this and just consider the magnitude as we are already aware of the direction, hence why I've not included the negative sign above.

A natural question to ask is: what is the significance of RC? It has units of time! RC is what is known as the **time constant**.

Definition 0.16 (Time Constant) — The time it takes for the current/voltage/charge to decrease by a factor of e ($\approx 37\%$).

Example 0.17 Find a function of RC that describes the 'half life' of a capacitor.

Solution. The half life $t_{\frac{1}{2}}$ is the time it takes for our current/voltage/charge to halve *i.e.* $\frac{I_0}{I}$ or $\frac{V_0}{V}$ or $\frac{Q_0}{Q}$ is equal to $\frac{1}{2}$.

$$\therefore\ e^{-\frac{t_{1/2}}{RC}} = \frac{1}{2} \Rightarrow \boxed{t_{\frac{1}{2}} = RC\ln 2}$$

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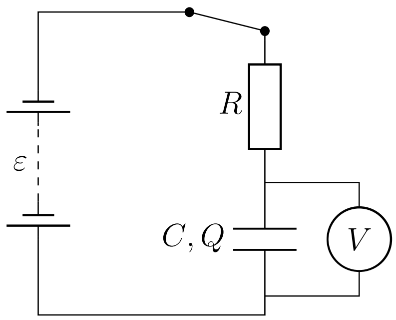
Charging Capacitor

Again, we proceed with a similar argument for the **capacitor charging**.



By Kirchoff's second law, $V + V_R = \varepsilon \Rightarrow \frac{Q}{C} + R\frac{\mathrm{d}Q}{\mathrm{d}t} = \varepsilon$.

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\varepsilon - \frac{Q}{C}}{R} = \frac{\varepsilon C - Q}{RC}$$

$$\therefore\ \int_0^Q \frac{\mathrm{d}Q}{\varepsilon C - Q} = \frac{1}{RC}\int_0^t \mathrm{d}t$$

Remark 0.18 The limits should be fairly clear that initially the capacitor has 0C charge.

$$-\left[\ln|\varepsilon C - Q|\right]_0^Q = \frac{t}{RC}$$

$$\ln\left(\frac{\varepsilon C - Q}{\varepsilon C}\right) = -\frac{t}{RC}$$

$$1 - \frac{Q}{\varepsilon C} = e^{-\frac{t}{RC}}$$

$$Q = \varepsilon C\left(1 - e^{-\frac{t}{RC}}\right) = \boxed{Q_f\left(1 - e^{-\frac{t}{RC}}\right)}$$

where Q_f is the final charge. For the voltage, dividing by C gives $V = \varepsilon\left(1 - e^{-\frac{t}{RC}}\right)$. Again, for current we can differentiate using the definition $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$.

$$I = \frac{Q_f}{RC}e^{-\frac{t}{RC}} = \frac{\varepsilon}{R}e^{-\frac{t}{RC}} = \boxed{I_f e^{-\frac{t}{RC}}}$$

Formula 0.19 (Charging of Capacitors - Charge, Voltage and Current)

$$Q = Q_f\left(1 - e^{-\frac{t}{RC}}\right)$$
$$V = \varepsilon\left(1 - e^{-\frac{t}{RC}}\right)$$
$$I = I_f e^{-\frac{t}{RC}}$$

We can graph these, and below are example graphs of charging (left) and discharging (right) for voltage against time.